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Nonperturbative Recursion Relations in $\mathcal{N} = 2$
Supersymmetric Gauge Theories

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This Thesis Is Dedicated To
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ABSTRACT OF THE DISSERTATION

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by

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Linear recursion relations for the instanton corrections to the effective prepotential are derived for two cases of $\mathcal{N} = 2$ supersymmetric gauge theories; the first case with an arbitrary number of hypermultiplets in the fundamental representation of an arbitrary classical gauge group, and the second case with one hypermultiplet in the adjoint representation of $SU(N)$. The construction for both cases proceed from the Seiberg-Witten solutions and the renormalization group type equations for the prepotential. Successive iterations of these recursion relations allow us to simply obtain instanton corrections to an arbitrarily high order. Checks with previous results in the literature were performed. Other theoretical properties and generalizations are also discussed.

Chapter 1

Introduction

Over the past half decade there has been great progress in understanding the non-perturbative dynamics of $\mathcal{N}=2$ SUSY gauge theories starting with the $SU(2)$ case [1][2], with further generalizations to other gauge groups [3]-[11] with additions of matter hypermultiplets [12]-[14]. Non-perturbative corrections in weak coupling corresponding to instanton effects [15] were evaluated using field theory techniques to one instanton [15][16][17][18] and two instanton [19][20] orders. Some of the previous instanton calculations using the Seiberg-Witten ansatz were performed by solving the Picard-Fuchs equations for the period integrals corresponding to the quantum moduli parameters representing the set of vacuum expectation values of the Higgs fields [21][36][37][38][39]. Other previous calculations involved solving the period integrals directly [22], and were found to be in agreement with the Picard-Fuchs and field theory results. In an intriguing paper [23], a recursion relation for the instanton corrections to the effective prepotential \mathcal{F} was found for the pure $SU(2)$ case which led us to seek a generalization of this result for any gauge group and number of matter hypermultiplets.

In a related development, the Seiberg-Witten equations were viewed analo-

gously to the Whitham hierarchy equations [24]-[28] and the WDVV equations [29][30]. Nonlinear recursion relations for the instanton corrections involving Jacobi θ -functions (which themselves involve τ_{ij} as in [35]) were derived starting from the Whitham hierarchy equations [31]-[33]. The beta function of the prepotential \mathcal{F} , first observed in [34] and later proved in [35], provides a very direct way at calculating instanton corrections to the prepotential \mathcal{F} without having to perform complicated hyperelliptic integrals and immediately obtaining rational expressions without θ -functions. The Seiberg-Witten solutions for classical gauge groups $SU(N)$, $SO(N)$, and $Sp(N)$ with matter hypermultiplets in the fundamental representation of the gauge group and the renormalization group like equation for the prepotential \mathcal{F} , led us to the discovery of a general recursion relation expressing the n -th order instanton correction to the prepotential \mathcal{F} in terms of the $(n-1)th, \dots, 1st$ order instanton corrections [41].

In a further development, connections between Seiberg-Witten theory and integrable systems were discovered first for the case of pure $\mathcal{N}=2$ super Yang-Mills theory in connection with Toda lattices [9] and Whitham theory [24]-[28]. Later connections between $\mathcal{N}=2$ super Yang-Mills theory with one hypermultiplet in the adjoint representation of the gauge group, and the Hitchin [42] and Calogero-Moser [43] integrable systems was made. (There are claims that the Calogero-Moser integrable system can be derived from the Hitchin integrable system [44]). Convenient parameterizations of the Calogero-Moser integrable system useful for performing explicit Seiberg-Witten type of calculations were discovered [45]. The Calogero-Moser construction [45] of the Seiberg-Witten solution for $\mathcal{N}=2$ super Yang-Mills theory with one hypermultiplet in the adjoint representation of the gauge group $SU(N)$ and the renormalization group like equation for the prepotential \mathcal{F} , led us to the discovery of a general recursion relation expressing the n -th order instanton

correction to the prepotential \mathcal{F} in terms of the (n-1)-th, ..., first order instanton corrections [56].

We start off this thesis by reviewing the original Seiberg-Witten problem for $\mathcal{N}=2$ super Yang-Mills theory with an $SU(2)$ gauge group with no hypermultiplets. Then we will explore extensions to the cases with hypermultiplets in the fundamental representation of any arbitrary classical gauge group, along with explicit calculations for several special $SU(2)$ and $SU(3)$ cases. Lastly we will discuss the case of an adjoint hypermultiplet with gauge group $SU(N)$, along with explicit calculations for the $SU(2)$ case and discussions of S-duality. A summary of the recursion relations methodology in Seiberg-Witten theory and possible generalizations are discussed in the conclusion.

Chapter 2

Seiberg-Witten Theory

The Seiberg-Witten (SW) ansatz gives a prescription for determining the prepotential of the effective action for $\mathcal{N} = 2$ supersymmetric Yang-Mills gauge theories, as well as for determining the spectrum of BPS states [1].

2.1. $\mathcal{N} = 2$ Supersymmetric Gauge Theories

$\mathcal{N} = 2$ supersymmetric gauge theories are constructed out of an $\mathcal{N} = 2$ chiral multiplet with an optional $\mathcal{N} = 2$ hypermultiplet. In terms $\mathcal{N} = 1$ chiral and vector multiplets, the $\mathcal{N} = 2$ chiral multiplet consist of one $\mathcal{N} = 1$ vector multiplet and one $\mathcal{N} = 1$ chiral multiplet all in the adjoint representation of the gauge group with the following $\mathcal{N} = 1$ superfield content:

$$\begin{aligned} W_\alpha &\rightarrow (A_\mu, \lambda, D) \\ \Phi &\rightarrow (\phi, \psi, F) \end{aligned} \tag{2.1}$$

where A_μ is a gauge boson, λ, ψ are Weyl fermions, ϕ is a scalar field, and D, F are auxiliary fields. The $\mathcal{N} = 2$ hypermultiplet consists of two $\mathcal{N} = 1$ chiral multiplets

all in a representations of the gauge group with the following $\mathcal{N} = 1$ superfield content:

$$\begin{aligned} Q &\rightarrow (\psi_q, q, F) \\ \tilde{Q} &\rightarrow (\psi_{\tilde{q}}^\dagger, \tilde{q}^\dagger, \tilde{F}) \end{aligned} \quad (2.2)$$

where $\psi_q, \psi_{\tilde{q}}^\dagger$ are Weyl fermions, q, \tilde{q}^\dagger form a complex scalar, and F, \tilde{F} are auxillary fields.

The most general renormalizable $\mathcal{N} = 1$ action consistent with the $\mathcal{N} = 2$ supersymmetry field content in $\mathcal{N} = 1$ superfield notation takes on the form (in Weinberg's notation [57])

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{chiral} + \mathcal{L}_{hyper} \\ \mathcal{L}_{chiral} &= \frac{1}{2} \Phi^\dagger e^{-2V} \Phi |_D - \frac{1}{2} \left[\frac{\tau}{8\pi i} W_\alpha W^\alpha |_F + h.c. \right] \\ \mathcal{L}_{hyper} &= \sum_{i=1}^{N_f} \frac{1}{2} \{ [Q_i^\dagger e^{-2V} Q_i + \tilde{Q}_i e^{2V} \tilde{Q}_i^\dagger] |_D + [\tilde{Q}_i \Phi Q_i + m_i \tilde{Q}_i Q_i] |_F + h.c. \} \end{aligned} \quad (2.3)$$

where the D and F integrations are over $d^4\theta$ and $d^2\theta$ respectively, and

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \quad (2.4)$$

is the complexified gauge coupling constant. In component form, the bosonic part of the action with the auxillary fields eliminated using the Euler-Lagrange equations of motion will take on the form

$$\begin{aligned} \mathcal{L} &= (D_\mu \phi)^\dagger D^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{g^2 \theta}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - V(\phi, \phi^\dagger) \\ &+ (interactions) \end{aligned} \quad (2.5)$$

where

$$V(\phi, \phi^\dagger) = \frac{1}{2} \text{tr}([\phi^\dagger, \phi])^2 \quad (2.6)$$

To minimize the potential $V(\phi, \phi^\dagger)$ without setting all the expectation values of all the scalar fields to zero, one can perform a spontaneous symmetry breaking of the gauge symmetry by keeping the scalar fields in the Cartan subalgebra of the gauge group and giving them a non-zero expectation value, while all the other scalar fields are given a zero expectation value. (For an $SU(N_c)$ gauge group, this breaks the gauge symmetry down to a $U(1)^{N_c-1}$ Coulomb phase). This process gives masses to the fields outside of the Cartan subalgebra and breaks $\mathcal{N} = 2$ supersymmetry. In order to preserve $\mathcal{N} = 2$ supersymmetry, one integrates out all the massive fields outside of the Cartan subalgebra, which will restore $\mathcal{N} = 2$ supersymmetry in the remaining Cartan subalgebra fields but will also add in non-renormalizable terms consistent with the other symmetries in the theory. Additionally, an anomaly along with this spontaneous symmetry breaking and subsequent integrating out prescription, breaks the classical \mathcal{R} -symmetry group of the theory to a subgroup of the original classical \mathcal{R} -symmetry [1] [57].

The most general non-renormalizable $\mathcal{N} = 1$ supersymmetric action consistent with the $\mathcal{N} = 2$ chiral multiplet field content in $\mathcal{N} = 1$ superfield notation takes on the form

$$\mathcal{L}_{chiral} = \frac{1}{2} [K(\Phi, \Phi^\dagger e^{-2V})] |_D - \frac{1}{2} [h(\Phi) W_\alpha W_\alpha |_F + h.c.] \quad (2.7)$$

where $h(\Phi)$ is a holomorphic coupling that is equal to $\frac{\tau}{4\pi i}$ at tree level. In general, a 1PI effective action of this form will have non-holomorphic contributions to τ

coming from the IR if there are massless particles in the theory [16]. One way around this is to take the Wilson effective action where all massive fields with mass greater than a scale a ¹ are integrated out and in the loop integrals, the region of integration is over momenta $\Lambda < |k| < a$ with $\Lambda \ll a$, where Λ is the scale generated from dimensional transmutation like an IR cutoff while a acts like a UV cutoff. This Wilson effective action preserves the holomorphicity of the coupling constant τ [16].

Imposing $\mathcal{N} = 2$ supersymmetry explicitly requires

$$K(\Phi, \Phi^\dagger) = \text{Im}[\Phi^\dagger \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi}] \quad , \quad h_{AB}(\Phi) = \frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi_A \partial \Phi_B} \quad (2.8)$$

producing a Wilson effective action of the form

$$\mathcal{L}_{chiral} = \frac{1}{8\pi} \text{Im} \{ [\frac{\partial \mathcal{F}(A)}{\partial A^i} \bar{A}^i] |_D + \frac{1}{2} [\frac{\partial^2 \mathcal{F}(A)}{\partial A^i \partial A^j} W_i W_j] |_F \} \quad (2.9)$$

where \mathcal{F} is a holomorphic prepotential that is the object of interest for which Seiberg-Witten [1] provided a prescription for calculating it.

2.2. Seiberg-Witten ansatz for pure $SU(2)$ super Yang-Mills Theory

Previous nonperturbative calculations performed in SUSY gauge theories using field theory methods [15][16] were done as weak coupling expansions around instanton solutions, which contributed as corrections to the prepotential \mathcal{F} for the $\mathcal{N} = 2$ SUSY cases.

For an $\mathcal{N} = 2$ super Yang-Mills theory with gauge group $SU(2)$ and no hy-

¹In general, a is taken to be smallest of the expectation values of the Higgs scalar fields in the Cartan subalgebra. The masses given to the massless fields from spontaneous symmetry breaking are of the order of the expectation value of the Higgs scalar fields in the Cartan subalgebra.

permultiplets that's spontaneously broken as prescribed in the last section, the prepotential in the Wilson effective action will have the general form

$$\mathcal{F}(a) = \frac{i}{2\pi} a^2 \ln\left(\frac{a^2}{\Lambda}\right) + \sum_{j=1}^{\infty} b_m \left(\frac{\Lambda}{a}\right)^{4m} \quad (2.10)$$

where a is the expectation value of the remaining Higgs field left that's in the Cartan subalgebra of $SU(2)$, and Λ is the scale produced by the one-loop quantum corrections via renormalization. The first term in (2.10) are the contributions from the tree level and one-loop terms, while the second sum consists of the sum of instanton factors

$$\left(\frac{\Lambda}{a}\right)^{4m} = e^{m2\pi i\tau} \quad (2.11)$$

weighted by b_m 's, due to the $m - th$ instanton. τ is the complexified coupling constant (2.4).

Seiberg-Witten in [1] show that the instanton sum in (2.10) is indeed nonvanishing, and uses that fact to derive a prescription to exactly solve for the b_m 's without using field theory methods. To show this, one starts off by defining a Kahler metric on the moduli space

$$\begin{aligned} ds^2 &= Im[\tau da d\bar{a}] \\ &= Im[da_D d\bar{a}] \end{aligned} \quad (2.12)$$

where

$$\tau = \frac{\partial^2 \mathcal{F}}{\partial a^2} \quad , \quad a_D = \frac{\partial \mathcal{F}}{\partial a} \quad (2.13)$$

Taking the first term in (2.10) and substituting it into (2.13), we get

$$a_D = \frac{2ia}{\pi} \ln\left(\frac{a}{\Lambda}\right) + \frac{ia}{\pi} \quad (2.14)$$

which is representative of the high energy perturbative part of the asymptotically free theory. A change of variables $u = \frac{1}{2}a^2$ is performed to identify in the moduli space the gauge equivalent moduli $a \rightarrow -a$ via the Weyl symmetry of $SU(2)$. The moduli space represented by the perturbative part of the theory is the region $u \rightarrow \infty$, where under a monodromy at large $u \rightarrow \infty$ produces

$$\ln(a) \rightarrow \ln(a) + \pi i \quad (2.15)$$

$$\ln(u) \rightarrow \ln(u) + 2\pi i$$

which induces a change in the moduli

$$a_D \rightarrow -a_D + 2a \quad (2.16)$$

$$a \rightarrow -a$$

using (2.14). Representing the moduli as a vector

$$v = \begin{pmatrix} a_D \\ a \end{pmatrix} \quad (2.17)$$

the moduli change (2.16) has the form

$$v \rightarrow M_\infty v \quad , \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \quad (2.18)$$

There mere fact that M_∞ is not the identity, implies the existence of singularities in the moduli space. If the metric on the moduli space $Im[\tau(a)]$ is globally defined it won't be positive definite because the harmonic function $Im[\tau(a)]$ can't have a minimum. Hence the metric can only be defined locally at these singularities in the moduli space.

A closer examination of the moduli space metric (2.13) reveals that it's invariant under $v \rightarrow Mv$ transformations, if $M \in SL(2, R)$. In general, $SL(2, R)$ is generated by the basis

$$T_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} , \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.19)$$

Under a T_b transformation, the moduli transforms as

$$\begin{aligned} a_D &\rightarrow a_D + ba \\ a &\rightarrow a \end{aligned} \quad (2.20)$$

which corresponds to a transformation of $\theta \rightarrow \theta + 2\pi b$ for the θ angle in the Lagrangian (2.9). For invariance of (2.9) under T_b , we require $b \in Z$ and hence the moduli transformations are invariant under $SL(2, Z)$.

Under an S transformation, the moduli transforms as

$$\begin{aligned} a_D &\rightarrow a \\ a &\rightarrow -a_D \end{aligned} \quad (2.21)$$

which is realized in the magnetic sector of the theory, where a magnetic monopole is coupled to a dual magnetic vector field. The equivalent Wilson effective action

was derived [1] to be

$$\mathcal{L}_{chiral} = \frac{1}{8\pi} \text{Im} \left\{ \left[\frac{\partial \mathcal{F}_D(A_D)}{\partial A_D} \bar{A}_D \right] |_D + \frac{1}{2} \left[\frac{\partial^2 \mathcal{F}_D(A_D)}{\partial A_D^2} W_D^2 \right] |_F \right\} \quad (2.22)$$

where

$$\begin{aligned} \frac{\partial^2 \mathcal{F}_D(A_D)}{\partial A_D^2} &= \tau_D \\ &= -\frac{1}{\tau} \end{aligned} \quad (2.23)$$

Equation (2.22) is like a “dual” magnetic theory coupled to magnetic monopoles, with a coupling constant $\tau_D = -\frac{1}{\tau}$ exhibiting the S transformation of the original Lagrangian (2.9).

These $SL(2, Z)$ transformations are in general not a “symmetry” but a way of transforming between different local descriptions of the moduli space. A fundamental representation of $SL(2, Z)$ will be a basis of monodromy matrices including M_∞ . If the monodromy matrices representing the singularities commuted with M_∞ , then that would make a^2 a good global coordinate with a global harmonic function $\text{Im}[\tau(a)]$ representing the moduli space. But previously it was argued that this metric will not be positive definite. Hence the other option is that the monodromy matrices do not commute with M_∞ , and will form a non-abelian representation for $SL(2, Z)$.

In the original $SU(2)$ $\mathcal{N} = 2$ super Yang-Mills theory, there’s a classical $U(1)_{\mathcal{R}}$ \mathcal{R} -symmetry that gets broken down to Z_8 by an anomaly. Under the spontaneous symmetry breaking and integrating out prescription outlined in the previous section, the Z_8 gets further broken down to Z_4 . Under the change of coordinates $u = \frac{1}{2}a^2$, there is a Z_2 \mathcal{R} -symmetry remaining. In a minimal non-abelian representation

of $SL(2, Z)$, two singularities in the u -plane of the moduli space “related” to one another by $u \rightarrow -u$ along with the singularity at infinity are conjectured to produce a fundamental representation of for $SL(2, Z)$. Seiberg-Witten proposed [1] that the two strong coupling singularities in the u -plane of the moduli space are that of the massless charged particle BPS states.

In $SU(2)$ supersymmetric Yang-Mills theory, in general there are electrically and/or magnetically charged particles of mass M and charge Z defined as

$$Z = n_m a_D + n_e a \quad (2.24)$$

$$M \geq \sqrt{2}|Z| \quad (2.25)$$

where n_m and n_e are the number of magnetic and electric charges respectively. The BPS states saturate the bound $M = \sqrt{2}|Z|$.

For the massless magnetic monopole, it will have charges $(n_m, n_e) = (1, 0)$ in a region around $a_D = 0$ in the moduli space with a mass $M = 0$. Taking $u = 1$ as the point in the moduli space for the massless monopole, the surrounding region around it can be parameterized as $a_D \approx c(u - 1)$ where c is a constant. In the dual magnetic sector of the theory (2.22), the magnetic coupling constant at tree level will be

$$\begin{aligned} \tau_D &= \frac{\partial^2 \mathcal{F}_D(a_D)}{\partial a_D^2} \\ &\approx -\frac{i}{\pi} \ln(a_D) \end{aligned} \quad (2.26)$$

which integrates to

$$a = \frac{\partial \mathcal{F}_D(a_D)}{\partial a_D} \quad (2.27)$$

$$\begin{aligned}
&\approx a_0 + \frac{i}{\pi} a_D \ln(a_D) \\
&\approx a_0 + \frac{i}{\pi} c(u-1) \ln[c(u-1)]
\end{aligned}$$

where a_0 is a constant. A monodromy around $u = 1$ will transform as

$$\ln[c(u-1)] \rightarrow \ln[c(u-1)] + 2\pi i \quad (2.28)$$

$$a_D \rightarrow a_D \quad (2.29)$$

$$a \rightarrow a - 2a_D$$

where in terms of the moduli space vector v (2.17)

$$v \rightarrow M_1 v \quad , \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad (2.30)$$

If we take the charges of particle as a row vector

$$q = (n_m, n_e) \quad (2.31)$$

for which the massless magnetic monopole will have $q_1 = (1, 0)$, the monodromy matrix M_1 will have an invariant transformation $q_1 M_1 = q_1$ with the charge vector q_1 .

To examine what the third singularity is, the monodromy at $u \rightarrow \infty$ can be decomposed into the monodromies around the $u = 1$ and the third singularity denoted as $u = -1$. In terms of the monodromy matrices, it will take on the form

$M_\infty = M_1 M_{-1}$ which produces

$$M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \quad (2.32)$$

In terms of charges represented by (2.31), we get a massless dyon $q_{-1} = (1, -1)$ being invariant under $q_{-1} M_{-1} = q_{-1}$.

Hence the monodromy matrices M_∞, M_1, M_{-1} form a subgroup $\Gamma(2)$ of $SL(2, Z)$. The solution to this math problem is in [58] and used by Seiberg-Witten [1] as

$$a = \oint_A d\lambda \quad , \quad a_D = \oint_B d\lambda$$

$$d\lambda = \frac{\sqrt{2}}{2\pi} \frac{(x-u)dx}{y} \quad , \quad y^2 = (x-1)(x+1)(x-u) \quad (2.33)$$

where the contour integrals are over a Riemann surface with branch cuts between -1 and 1 on the real x-axis and between u to ∞ in the complex x-plane. The integral for a has a closed contour A taken around the branch cut between $x=-1$ and $x=1$, while the integral for a_D has a closed contour B taken through the branch cuts at $x=1$ and $x=u$. (The topology of this Riemann surface is a torus with the A and B contours taken as the homology cycles, where the B contour is around the center hole of the torus [58]). These integrals can be solved by various means including the Picard-Fuchs equations [21], rewriting the equations in terms of a recursion relation [23], or by solving the integrals directly [40].

Using similar types of arguments, these results were generalized to other gauge groups and additions of matter hypermultiplets [2]-[14]. Various other methods of solving the resulting integrals in these generalizations were found [22]-[41][48][49][56].

Chapter 3

SW theory with fundamental hypermultiplets

3.1. The Seiberg-Witten Solution for Arbitrary Classical Gauge Group G

We consider, $\mathcal{N}=2$ SUSY gauge theories with classical gauge groups $SU(r+1)$, $SO(2r+1)$, $Sp(2r)$ and $SO(2r)$, of rank r and number of colours $N_c = r+1$, $2r+1$, $2r$, and $2r$ respectively. We include N_f hypermultiplets in the fundamental representation of the gauge group, with bare masses m_j , $j = 1, \dots, N_f$. We restrict to the asymptotically free theories; this limits the hypermultiplet contents N_f . (ie. $N_f < 2N_c$ for $SU(N_c)$). The classical vacuum expectation value of the gauge scalar ϕ is parameterized by complex moduli $\bar{a}_k, k = 1, \dots, r$ as follows

$$\begin{aligned}
SU(r+1) \quad \phi &= \text{diag}[\bar{a}_1, \dots, \bar{a}_r, \bar{a}_{r+1}] & \bar{a}_1 + \dots + \bar{a}_r + \bar{a}_{r+1} &= 0 \\
SO(2r+1) \quad \phi &= \text{diag}[\mathcal{A}_1, \dots, \mathcal{A}_r, 0] \\
Sp(2r) \quad \phi &= \text{diag}[\bar{a}_1, -\bar{a}_1, \dots, \bar{a}_r, -\bar{a}_r] \\
SO(2r) \quad \phi &= \text{diag}[\mathcal{A}_1, \dots, \mathcal{A}_r] & \mathcal{A}_k &= \begin{pmatrix} 0 & \bar{a}_k \\ -\bar{a}_k & 0 \end{pmatrix}
\end{aligned} \tag{3.1}$$

For generic \bar{a}_k 's, the gauge symmetry is broken to $U(1)^r$ and the dynamics is that of an Abelian Coulomb phase. The Wilson effective Lagrangian of the quantum theory to leading order in the low momentum expansion in the Abelian Coulomb phase is completely characterized by a complex analytic prepotential $\mathcal{F}(a)$.

The SW ansatz for determining the full prepotential \mathcal{F} is based on a choice of a fibration of spectral curves over the space of vacua, and of a meromorphic 1-form $d\lambda$ on each of these curves. The renormalized order parameters a_k of the theory, their duals $a_{D,k}$, and the prepotential \mathcal{F} are given by

$$2\pi i a_k = \oint_{A_k} d\lambda, \quad 2\pi i a_{D,k} = \oint_{B_k} d\lambda, \quad a_{D,k} = \frac{\partial \mathcal{F}}{\partial a_k} \tag{3.2}$$

with A_k, B_k a suitable set of homology cycles on the spectral curves [40].

For all $\mathcal{N}=2$ supersymmetric gauge theories based on classical gauge groups with N_f hypermultiplets in the fundamental representation of the gauge group, the spectral curves and meromorphic 1-forms are

$$\begin{aligned}
y^2 &= A^2(x) - B(x) \\
d\lambda &= \frac{x}{y} \left(A' - \frac{AB'}{2B} \right) dx
\end{aligned} \tag{3.3}$$

where

$$\left. \begin{array}{l} SU(r+1) \\ SO(2r+1) \\ Sp(2r)^1 \\ SO(2r) \end{array} \right\} \begin{array}{l} A(x) = \prod_{k=1}^{r+1} (x - \bar{a}_k) \quad B(x) = \bar{\Lambda}^2 \prod_{j=1}^{N_f} (x + m_j) \\ A(x) = x^a \prod_{k=1}^r (x^2 - \bar{a}_k^2) \quad B(x) = \bar{\Lambda}^2 x^b \prod_{j=1}^{N_f} (x^2 - m_j^2) \end{array} \quad (3.4)$$

with $\bar{\Lambda} \equiv \Lambda^q$

$$\begin{array}{llll} SU(r+1) & q = r+1 - N_f/2 & & \\ SO(2r+1) & q = 2r-1 - N_f & a=0 & b=2 \\ Sp(2r) & q = 2r+2 - N_f & a=2 & b=0 \\ SO(2r) & q = 2r-2 - N_f & a=0 & b=4 \end{array} \quad (3.5)$$

respectively. The spectral curves (3.4) for $SO(2r+1)$, $Sp(2r)$ and $SO(2r)$ can be obtained from the $SU(2r)$ spectral curve by a suitable restriction on the classical moduli \bar{a}_k 's and masses [40].

For gauge theories with classical gauge groups and asymptotically free coupling obeying the constraint $q > 0$, general arguments based on the holomorphicity of \mathcal{F} , perturbative non-renormalization theorems beyond 1-loop order, the nature of instanton corrections, and restrictions of $U(1)_R$ invariance, constrain \mathcal{F} to have the form

$$\mathcal{F}(a) = \frac{2q}{\pi i} \sum_{i=1}^r a_i^2 + \frac{i}{4\pi} \left[\sum_{\alpha} (\alpha \cdot a)^2 \log \frac{(\alpha \cdot a)^2}{\Lambda^2} \right]$$

¹For simplicity, we restrict attention here to the $Sp(2r)$ case with at least two massless hypermultiplets. The cases with one or no massless hypermultiplets may be treated accordingly [40].

$$\begin{aligned}
& - \sum_i \sum_{j=1}^{N_f} (\lambda_i \cdot a + m_j)^2 \log \frac{(\lambda_i \cdot a + m_j)^2}{\Lambda^2} \Big] \\
& + \sum_{m=1}^{\infty} \frac{\Lambda^{2mq}}{2m\pi i} \mathcal{F}^{(m)}(a)
\end{aligned} \tag{3.6}$$

where $\lambda_i = \pm e_i$ for SO and Sp and $\lambda_i = e_i$ for SU in an orthonormal basis \vec{e}_i , and α are the roots of the gauge group G . (The $SU(r)$ solution requires an additional overall factor of $\frac{1}{2}$).

The terms on the right side are respectively the classical prepotential, the contribution of perturbative one-loop effects, and m -instanton processes contributions². Λ is the dynamically generated scale of the theory.

3.2. Renormalization Group Type Equations

In [35], a renormalization group type equation for the prepotential \mathcal{F} was derived using the SW ansatz equations (3.2)

$$\Lambda \frac{\partial \mathcal{F}}{\partial \Lambda} = \frac{q}{\pi i} \sum_{k=1}^r \bar{a}_k^2 \tag{3.7}$$

up to an additive term independent of a_k and \bar{a}_k which is physically immaterial. (The $SU(r)$ case requires an additional factor of $\frac{1}{2}$).

In [22] an efficient algorithm was presented for calculating the renormalized order parameters a_k and their duals $a_{D,k}$ in terms of the classical order parameters \bar{a}_k to any order of perturbation theory in a regime where $\bar{\Lambda}$ is small and the \bar{a}_k 's are well-separated. The calculation of a_k starts off from equations (3.2) and (3.3)

²The normalization of the instanton contributions in the present paper differs from that of [22][40] by a factor $\frac{1}{4m\pi i}$ for $SU(N_c)$ and $\frac{1}{2m\pi i}$ for $SO(2r+1)$, $Sp(2r)$, and $SO(2r)$. For our purposes, it will be convenient to use the normalization of (3.6).

producing a final result

$$a_k = \sum_{m=0}^{\infty} \bar{\Lambda}^{2m} \Delta_k^{(m)}(\bar{a}) \quad (3.8)$$

where we set $\Delta_k^{(0)}(\bar{a}) \equiv \bar{a}_k$, and we have

$$\Delta_k^{(m)}(\bar{a}) = \frac{1}{2^{2m}(m!)^2} \left(\frac{\partial}{\partial \bar{a}_k} \right)^{2m-1} S_k(\bar{a}_k, \bar{a})^m, \quad m \neq 0 \quad (3.9)$$

with

$$\begin{aligned} SU(r+1) \quad S_k(x, a) &= \frac{\prod_{j=1}^{N_f} (x + m_j)}{\prod_{l \neq k} (x - a_l)^2} \\ SO(2r+1) \quad S_k(x, a) &= \frac{x^2 \prod_{j=1}^{N_f} (x^2 - m_j^2)}{(x + a_k)^2 \prod_{l \neq k} (x^2 - a_l^2)^2} \\ Sp(2r)^1 \quad S_k(x, a) &= \frac{\prod_{j=1}^{N_f-2} (x^2 - m_j^2)}{(x + a_k)^2 \prod_{l \neq k} (x^2 - a_l^2)^2} \\ SO(2r) \quad S_k(x, a) &= \frac{x^4 \prod_{j=1}^{N_f} (x^2 - m_j^2)}{(x + a_k)^2 \prod_{l \neq k} (x^2 - a_l^2)^2} \end{aligned} \quad (3.10)$$

and $\bar{\Lambda}$ defined as previously (3.5).

Equations (3.7)(3.8)(3.9) (3.10) suffices to determine the prepotential \mathcal{F} in terms of the renormalized order parameters a_k order by order in powers of $\bar{\Lambda}^2$.

3.3. Recursion Relation for the Prepotential \mathcal{F}

A very direct way of deriving the form of the instanton corrections to the prepotential \mathcal{F} starts off from the beta function on the right hand side of (3.7). Substituting the ansatz for the prepotential (3.6) into the beta function (3.7), one obtains

$$\sum_{k=1}^r \bar{a}_k^2 = \sum_{k=1}^r a_k^2 + \sum_{m=1}^{\infty} \bar{\Lambda}^{2m} \mathcal{F}^{(m)}(a) \quad (3.11)$$

Substituting (3.8) into (3.11), one obtains

$$\begin{aligned}
0 = & \sum_{k=1}^r \left[\sum_{m=0}^{\infty} \bar{\Lambda}^{2m} \Delta_k^{(m)}(\bar{a}) \right]^2 - \sum_{k=1}^r (\Delta_k^{(0)}(\bar{a}))^2 \\
& + \sum_{m=1}^{\infty} \bar{\Lambda}^{2m} \mathcal{F}^{(m)} \left(\sum_{n=0}^{\infty} \bar{\Lambda}^{2n} \Delta_k^{(n)}(\bar{a}) \right)
\end{aligned} \tag{3.12}$$

Expanding in powers of $\bar{\Lambda}^2$ in the last term and replacing the \bar{a}_k 's with a_k 's, the m -th order instanton correction to the prepotential \mathcal{F} takes on the form

$$\begin{aligned}
-\mathcal{F}^{(m)}(a) = & \sum_{k=1}^r \left[\sum_{\substack{i,j=0 \\ i+j=m}}^m \Delta_k^{(i)}(a) \Delta_k^{(j)}(a) \right] \\
& + \sum_{n=1}^{m-1} \frac{1}{n!} \sum_{\substack{\beta_1, \dots, \beta_{n+1}=1 \\ \beta_1 + \dots + \beta_{n+1} = m}}^{n-1} \sum_{\alpha_1, \dots, \alpha_n=1}^r \left[\prod_{i=1}^n \Delta_{\alpha_i}^{(\beta_i)}(a) \right] \left(\prod_{j=1}^n \frac{\partial}{\partial a_{\alpha_j}} \right) \mathcal{F}^{(\beta_{n+1})}(a)
\end{aligned} \tag{3.13}$$

which is a linear recursion relation for $\mathcal{F}^{(m)}(a)$ in terms of the lower order instanton corrections $\mathcal{F}^{(m-1)}(a), \dots, \mathcal{F}^{(1)}(a)$.

The intriguing part about the recursion relation (3.13) for $\mathcal{F}^{(n)}(a)$ is that it is linear in $\mathcal{F}^{(n-1)}(a), \dots, \mathcal{F}^{(1)}(a)$ and is valid for all classical gauge groups with the number of hypermultiplets in the fundamental representation constrained by $q > 0$. Previous recursion relations [23] were only valid for $SU(2)$ with no hypermultiplets and were non-linear.

3.4. Instanton Expansion of the Prepotential \mathcal{F}

Order by order in powers of $\bar{\Lambda}^2$, the first six instanton corrections (3.13) to the prepotential \mathcal{F} are

$$-\mathcal{F}^{(1)}(a) = \sum_{k=1}^r 2\Delta_k^{(0)}(a)\Delta_k^{(1)}(a) \quad (3.14)$$

$$-\mathcal{F}^{(2)}(a) = \sum_{k=1}^r \left[2\Delta_k^{(0)}(a)\Delta_k^{(2)}(a) + (\Delta_k^{(1)}(a))^2 \right] + \sum_{k=1}^r \Delta_k^{(1)}(a) \frac{\partial \mathcal{F}^{(1)}(a)}{\partial a_k} \quad (3.15)$$

$$\begin{aligned} -\mathcal{F}^{(3)}(a) &= \sum_{k=1}^r \left[2\Delta_k^{(0)}(a)\Delta_k^{(3)}(a) + 2\Delta_k^{(1)}(a)\Delta_k^{(2)}(a) \right] \\ &+ \sum_{k=1}^r \left[\Delta_k^{(1)}(a) \frac{\partial \mathcal{F}^{(2)}(a)}{\partial a_k} + \Delta_k^{(2)}(a) \frac{\partial \mathcal{F}^{(1)}(a)}{\partial a_k} \right] \\ &+ \frac{1}{2!} \sum_{k,m=1}^r \Delta_k^{(1)}(a)\Delta_m^{(1)}(a) \frac{\partial^2 \mathcal{F}^{(1)}(a)}{\partial a_k \partial a_m} \end{aligned} \quad (3.16)$$

$$\begin{aligned} -\mathcal{F}^{(4)}(a) &= \sum_{k=1}^r \left[2\Delta_k^{(0)}(a)\Delta_k^{(4)}(a) + 2\Delta_k^{(1)}(a)\Delta_k^{(3)}(a) + (\Delta_k^{(2)}(a))^2 \right] \\ &+ \sum_{k=1}^r \left[\Delta_k^{(1)}(a) \frac{\partial \mathcal{F}^{(3)}(a)}{\partial a_k} + \Delta_k^{(2)}(a) \frac{\partial \mathcal{F}^{(2)}(a)}{\partial a_k} + \Delta_k^{(3)}(a) \frac{\partial \mathcal{F}^{(1)}(a)}{\partial a_k} \right] \\ &+ \frac{1}{2!} \sum_{k,m=1}^r \left[\Delta_k^{(1)}(a)\Delta_m^{(1)}(a) \frac{\partial^2 \mathcal{F}^{(2)}(a)}{\partial a_k \partial a_m} + 2\Delta_k^{(1)}(a)\Delta_m^{(2)}(a) \frac{\partial^2 \mathcal{F}^{(1)}(a)}{\partial a_k \partial a_m} \right] \\ &+ \frac{1}{3!} \sum_{k,m,n=1}^r \Delta_k^{(1)}(a)\Delta_m^{(1)}(a)\Delta_n^{(1)}(a) \frac{\partial^3 \mathcal{F}^{(1)}(a)}{\partial a_k \partial a_m \partial a_n} \end{aligned} \quad (3.17)$$

$$\begin{aligned} -\mathcal{F}^{(5)}(a) &= \sum_{k=1}^r \left[2\Delta_k^{(0)}(a)\Delta_k^{(5)}(a) + 2\Delta_k^{(1)}(a)\Delta_k^{(4)}(a) + 2\Delta_k^{(2)}(a)\Delta_k^{(3)}(a) \right] \\ &+ \sum_{k=1}^r \left[\Delta_k^{(1)}(a) \frac{\partial \mathcal{F}^{(4)}(a)}{\partial a_k} + \Delta_k^{(2)}(a) \frac{\partial \mathcal{F}^{(3)}(a)}{\partial a_k} + \Delta_k^{(3)}(a) \frac{\partial \mathcal{F}^{(2)}(a)}{\partial a_k} \right. \\ &+ \left. \Delta_k^{(4)}(a) \frac{\partial \mathcal{F}^{(1)}(a)}{\partial a_k} \right] + \frac{1}{2!} \sum_{k,m=1}^r \left[\Delta_k^{(1)}(a)\Delta_m^{(1)}(a) \frac{\partial^2 \mathcal{F}^{(3)}(a)}{\partial a_k \partial a_m} \right. \\ &+ \left. 2\Delta_k^{(1)}(a)\Delta_m^{(3)}(a) \frac{\partial^2 \mathcal{F}^{(1)}(a)}{\partial a_k \partial a_m} \right. \\ &+ \left. \Delta_k^{(2)}(a)\Delta_m^{(2)}(a) \frac{\partial^2 \mathcal{F}^{(1)}(a)}{\partial a_k \partial a_m} + 2\Delta_k^{(1)}(a)\Delta_m^{(2)}(a) \frac{\partial^2 \mathcal{F}^{(2)}(a)}{\partial a_k \partial a_m} \right] \\ &+ \frac{1}{3!} \sum_{k,m,n=1}^r \left[\Delta_k^{(1)}(a)\Delta_m^{(1)}(a)\Delta_n^{(1)}(a) \frac{\partial^3 \mathcal{F}^{(2)}(a)}{\partial a_k \partial a_m \partial a_n} \right. \\ &+ \left. 3\Delta_k^{(2)}(a)\Delta_m^{(1)}(a)\Delta_n^{(1)}(a) \frac{\partial^3 \mathcal{F}^{(1)}(a)}{\partial a_k \partial a_m \partial a_n} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4!} \sum_{k,m,n,l=1}^r \Delta_k^{(1)}(a) \Delta_m^{(1)}(a) \Delta_n^{(1)}(a) \Delta_l^{(1)}(a) \frac{\partial^4 \mathcal{F}^{(2)}(a)}{\partial a_k \partial a_m \partial a_n \partial a_l} \\
-\mathcal{F}^{(6)}(a) & = \sum_{k=1}^r \left[2\Delta_k^{(0)}(a) \Delta_k^{(6)}(a) + 2\Delta_k^{(1)}(a) \Delta_k^{(5)}(a) + 2\Delta_k^{(2)}(a) \Delta_k^{(4)}(a) \right. \\
& + (\Delta_k^{(3)}(a))^2 \left. \right] + \sum_{k=1}^r \left[\Delta_k^{(1)}(a) \frac{\partial \mathcal{F}^{(5)}(a)}{\partial a_k} + \Delta_k^{(2)}(a) \frac{\partial \mathcal{F}^{(4)}(a)}{\partial a_k} \right. \\
& + \Delta_k^{(3)}(a) \frac{\partial \mathcal{F}^{(3)}(a)}{\partial a_k} + \Delta_k^{(4)}(a) \frac{\partial \mathcal{F}^{(2)}(a)}{\partial a_k} + \Delta_k^{(5)}(a) \frac{\partial \mathcal{F}^{(1)}(a)}{\partial a_k} \left. \right] \\
& + \frac{1}{2!} \sum_{k,m=1}^r \left[\Delta_k^{(1)}(a) \Delta_m^{(1)}(a) \frac{\partial^2 \mathcal{F}^{(4)}(a)}{\partial a_k \partial a_m} + 2\Delta_k^{(1)}(a) \Delta_m^{(4)}(a) \frac{\partial^2 \mathcal{F}^{(1)}(a)}{\partial a_k \partial a_m} \right. \\
& + \Delta_k^{(2)}(a) \Delta_m^{(2)}(a) \frac{\partial^2 \mathcal{F}^{(2)}(a)}{\partial a_k \partial a_m} + 2\Delta_k^{(1)}(a) \Delta_m^{(2)}(a) \frac{\partial^2 \mathcal{F}^{(3)}(a)}{\partial a_k \partial a_m} \\
& + 2\Delta_k^{(1)}(a) \Delta_m^{(3)}(a) \frac{\partial^2 \mathcal{F}^{(2)}(a)}{\partial a_k \partial a_m} + 2\Delta_k^{(2)}(a) \Delta_m^{(3)}(a) \frac{\partial^2 \mathcal{F}^{(1)}(a)}{\partial a_k \partial a_m} \left. \right] \\
& + \frac{1}{3!} \sum_{k,m,n=1}^r \left[\Delta_k^{(1)}(a) \Delta_m^{(1)}(a) \Delta_n^{(1)}(a) \frac{\partial^3 \mathcal{F}^{(3)}(a)}{\partial a_k \partial a_m \partial a_n} \right. \\
& + 3\Delta_k^{(1)}(a) \Delta_m^{(1)}(a) \Delta_n^{(3)}(a) \frac{\partial^3 \mathcal{F}^{(1)}(a)}{\partial a_k \partial a_m \partial a_n} \\
& + 3\Delta_k^{(1)}(a) \Delta_m^{(1)}(a) \Delta_n^{(2)}(a) \frac{\partial^3 \mathcal{F}^{(2)}(a)}{\partial a_k \partial a_m \partial a_n} \\
& + 3\Delta_k^{(1)}(a) \Delta_m^{(2)}(a) \Delta_n^{(2)}(a) \frac{\partial^3 \mathcal{F}^{(1)}(a)}{\partial a_k \partial a_m \partial a_n} \left. \right] \\
& + \frac{1}{4!} \sum_{k,m,n,l=1}^r \left[\Delta_k^{(1)}(a) \Delta_m^{(1)}(a) \Delta_n^{(1)}(a) \Delta_l^{(1)}(a) \frac{\partial^4 \mathcal{F}^{(2)}(a)}{\partial a_k \partial a_m \partial a_n \partial a_l} \right. \\
& + 4\Delta_k^{(2)}(a) \Delta_m^{(1)}(a) \Delta_n^{(1)}(a) \Delta_l^{(1)}(a) \frac{\partial^4 \mathcal{F}^{(1)}(a)}{\partial a_k \partial a_m \partial a_n \partial a_l} \left. \right] \\
& + \frac{1}{5!} \sum_{k,m,n,l,o=1}^r \Delta_k^{(1)}(a) \Delta_m^{(1)}(a) \Delta_n^{(1)}(a) \Delta_l^{(1)}(a) \Delta_o^{(1)}(a) \frac{\partial^5 \mathcal{F}^{(1)}(a)}{\partial a_k \partial a_m \partial a_n \partial a_l \partial a_o}
\end{aligned} \tag{3.18}$$

$$\tag{3.19}$$

A closer examination of the recursion relation (3.13) for the prepotential \mathcal{F} reveals that there is always a term of the form

$$2 \sum_{k=1}^r \Delta_k^{(0)}(a) \Delta_k^{(n)}(a) = 2 \sum_{k=1}^r a_k \Delta_k^{(n)}(a) \tag{3.20}$$

When performing explicit calculations for special cases of N_c and N_f , it is useful to rewrite terms of the form (3.20) so that there are no a_k 's sitting out in front. Using the definition (3.10) of $S_k(x, a)$ and performing contour integrals in the complex plane by residue methods as in [22], it can be shown that

$$2 \sum_{k=1}^r a_k \Delta_k^{(n)}(a) = -\frac{(2n-1)}{2^{2n-1}(n!)^2} \sum_{k=1}^r \left(\frac{\partial}{\partial a_k} \right)^{2n-2} S_k(a_k, a)^n \quad (3.21)$$

up to an a_k independent term that is physically immaterial for $q > 0$.

3.5. Comparison with Previous Results

In order to make explicit comparisons with results in the literature, the instanton corrections have to be rewritten in terms of symmetric polynomials in the a_k 's as follows.

For $SU(2)$, the existing results in the literature have the instanton expressions expressed in terms of

$$\begin{aligned} a_1 &= 2a \\ a_2 &= -2a \end{aligned} \quad (3.22)$$

Solving the recursion relation (3.13) for the pure $SU(2)$ case, the explicit form for the n -th order instanton correction to the prepotential \mathcal{F} was determined to be

$$\mathcal{F}^{(n)}(a) = \frac{1}{(2a)^{4n-2}} \sum_{j=1}^n \binom{4n-3}{j-1} \frac{(-1)^{j-1}}{j} \sum_{\substack{n_1, \dots, n_j=1 \\ n_1 + \dots + n_j = n}}^n b_{n_1} \dots b_{n_j} \quad (3.23)$$

where

$$b_n = \frac{(2n-3)!!}{(n!)^2} \quad (3.24)$$

which agrees with previous results [21][22]³.

Explicit evaluations for $N_f = 0, 1, 2, 3$ were performed, with $N_f = 3$ summarized here.

$$\begin{aligned} \mathcal{F}^{(1)} &= \frac{1}{2^2 a^2} [a^2(m_1 + m_2 + m_3) + m_1 m_2 m_3] \\ \mathcal{F}^{(2)} &= \frac{1}{2^8 a^6} [a^6 + a^4(m_1^2 + m_2^2 + m_3^2) - a^2(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2) + 5m_1^2 m_2^2 m_3^2] \\ \mathcal{F}^{(3)} &= \frac{m_1 m_2 m_3}{2^{11} a^{10}} [-3a^6 + 5a^4(m_1^2 + m_2^2 + m_3^2) - 7a^2(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2) \\ &\quad + 9m_1^2 m_2^2 m_3^2] \\ \mathcal{F}^{(4)} &= \frac{1}{2^{20} a^{14}} \{a^{12} - 6a^{10}(m_1^2 + m_2^2 + m_3^2) + a^8[5(m_1^4 + m_2^4 + m_3^4) \\ &\quad + 100(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2)] + a^6[1176m_1^2 m_2^2 m_3^2 \\ &\quad - 126(m_1^4 m_2^2 + m_1^2 m_2^4 + m_1^4 m_3^2 + m_1^2 m_3^4 + m_2^4 m_3^2 + m_2^2 m_3^4)] \\ &\quad + a^4[153(m_1^4 m_2^4 + m_1^4 m_3^4 + m_2^4 m_3^4) + 1332m_1^2 m_2^2 m_3^2(m_1^2 + m_2^2 + m_3^2)] \\ &\quad - 1430a^2 m_1^2 m_2^2 m_3^2(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2) + 1469m_1^4 m_2^4 m_3^4\} \\ \mathcal{F}^{(5)} &= \frac{m_1 m_2 m_3}{2^{23} a^{18}} \{35a^{12} - 210a^{10}(m_1^2 + m_2^2 + m_3^2) \\ &\quad + a^8[207(m_1^4 + m_2^4 + m_3^4) + 1260(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2)] \\ &\quad - 1210a^6(m_1^4 m_2^2 + m_1^2 m_2^4 + m_1^4 m_3^2 + m_1^2 m_3^4 + m_2^4 m_3^2 + m_2^2 m_3^4) \\ &\quad + a^4[1131(m_1^4 m_2^4 + m_1^4 m_3^4 + m_2^4 m_3^4) + 5960m_1^2 m_2^2 m_3^2(m_1^2 + m_2^2 + m_3^2)] \\ &\quad - 5250a^2 m_1^2 m_2^2 m_3^2(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2) + 4471m_1^4 m_2^4 m_3^4\} \\ \mathcal{F}^{(6)} &= \frac{1}{2^{29} a^{22}} \{5a^{16}(m_1^2 + m_2^2 + m_3^2) - a^{14}[210(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2) \\ &\quad + 14(m_1^4 + m_2^4 + m_3^4)] + a^{12}[9(m_1^6 + m_2^6 + m_3^6) + 6507m_1^2 m_2^2 m_3^2] \end{aligned}$$

³Our results agree exactly with those of [21] to eight instantons with the replacement $\Lambda^2 \rightarrow \frac{\Lambda^2}{2}$.

$$\begin{aligned}
& + 801(m_1^4 m_2^2 + m_1^2 m_2^4 + m_1^4 m_3^2 + m_1^2 m_3^4 + m_2^4 m_3^2 + m_2^2 m_3^4)] \\
& - a^{10}[660(m_1^6 m_2^2 + m_1^2 m_2^6 + m_1^6 m_3^2 + m_1^2 m_3^6 + m_2^6 m_3^2 + m_2^2 m_3^6) \\
& + 330(m_1^4 m_2^4 + m_1^4 m_3^4 + m_2^4 m_3^4) + 24420 m_1^2 m_2^2 m_3^2 (m_1^2 + m_2^2 + m_3^2)] \\
& + a^8[2769(m_1^6 m_2^4 + m_1^4 m_2^6 + m_1^6 m_3^4 + m_1^4 m_3^6 + m_2^6 m_3^4 + m_2^4 m_3^6) \\
& + m_1^2 m_2^2 m_3^2 (19851(m_1^4 + m_2^4 + m_3^4) + 87945(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2))] \\
& - a^6[295050 m_1^4 m_2^4 m_3^4 + 2310(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2) \\
& + 69510 m_1^2 m_2^2 m_3^2 (m_1^4 m_2^2 + m_1^2 m_2^4 + m_1^4 m_3^2 + m_1^2 m_3^4 + m_2^4 m_3^2 + m_2^2 m_3^4)] \\
& + a^4 m_1^2 m_2^2 m_3^2 [53839(m_1^4 m_2^4 + m_1^4 m_3^4 + m_2^4 m_3^4) \\
& + 224485 m_1^2 m_2^2 m_3^2 (m_1^2 + m_2^2 + m_3^2)] \\
& - 166896 a^2 m_1^4 m_2^4 m_3^4 (m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2) + 121191 m_1^6 m_2^6 m_3^6 \}
\end{aligned}$$

A check of the hypermultiplet decoupling limits of the $N_f = 3$ instanton corrections, by letting $\Lambda_3 m_3 = \Lambda_2^2$ and sending $m_3 \rightarrow \infty$, reproduces the $N_f = 2$ results. A further decoupling of a second hypermultiplet, by letting $\Lambda_2 m_2 = \Lambda_1^2$ and sending $m_c \rightarrow \infty$, reproduces the $N_f = 1$ results. Comparison with results in the literature [36][37][22] [31][32] show an agreement to four instantons up to a redefinition of the \bar{a}_k 's as discussed in [22].

For $SU(3)$, the existing results in the literature have the instanton corrections expressed in terms of the invariant $SU(3)$ symmetric polynomials u, v and the discriminant Δ

$$\begin{aligned}
u &= -a_1 a_2 - a_1 a_3 - a_2 a_3 \\
v &= a_1 a_2 a_3 \\
\Delta &= 4u^3 - 27v^3
\end{aligned} \tag{3.25}$$

and the p -th symmetric mass polynomials

$$t_p(m) = \sum_{j_1 < \dots < j_p} m_{j_1} \cdots m_{j_p} \quad (3.26)$$

Explicit evaluations for $N_f = 0, 1, 2, 3, 4, 5$ were performed, and are summarized here for $N_f = 0$.

$$\begin{aligned} \mathcal{F}^{(1)} &= \Lambda^6 \frac{3u}{\Delta} \\ \mathcal{F}^{(2)} &= \frac{\Lambda^{12}u}{16} \left[\frac{10935v^2}{\Delta^3} + \frac{153}{\Delta^2} \right] \\ \mathcal{F}^{(3)} &= \frac{3\Lambda^{18}u}{16} \left[\frac{4782969v^4}{2\Delta^5} + \frac{161109v^2}{2\Delta^4} + \frac{385}{\Delta^3} \right] \\ \mathcal{F}^{(4)} &= \frac{\Lambda^{24}u}{4096} \left[\frac{1707362095023v^6}{\Delta^7} + \frac{91216001799v^4}{\Delta^6} + \frac{1254600981v^2}{\Delta^5} \right. \\ &\quad \left. + \frac{3048885}{\Delta^4} \right] \\ \mathcal{F}^{(5)} &= \frac{5\Lambda^{30}u}{4096} \left[\frac{3788227372819653v^8}{10\Delta^9} + \frac{277223767370307v^6}{10\Delta^8} \right. \\ &\quad \left. + \frac{6447389599341v^4}{10\Delta^7} + \frac{50110037721v^2}{10\Delta^6} + \frac{7400133}{\Delta^5} \right] \\ \mathcal{F}^{(6)} &= \frac{3\Lambda^{36}u}{65535} \left[\frac{24952152189682606959v^{10}}{2\Delta^{11}} + \frac{2319087386959542567v^8}{2\Delta^{10}} \right. \\ &\quad \left. + \frac{38185135433846901v^6}{\Delta^9} + \frac{525166021552761v^4}{\Delta^8} \right. \\ &\quad \left. + \frac{5323867298775v^2}{2\Delta^7} + \frac{5295230391}{2\Delta^6} \right] \end{aligned}$$

A check of successive hypermultiplet decoupling limits of the $N_f = 5$ instanton corrections reproduces all of the $N_f < 5$ cases accordingly. Comparison with results in the literature [21][22][38][39] [31][32] show an agreement to three instantons up to a redefinition of the \bar{a}_k 's as discussed in [22].

3.6. Classical Moduli in Terms of Quantum Moduli

Another way of evaluating the beta function (3.7) of the prepotential \mathcal{F} involves inverting (3.8) to get

$$\bar{a}_k \equiv a_k + \sum_{m=1}^{\infty} \bar{\Lambda}^{2m} \beta_k^{(m)}(a) \quad (3.27)$$

where the $\beta_k(a)$'s are functions of the renormalized order parameters a_k .

A very direct way of deriving the form of the $\beta_k(a)$'s involves starting off with (3.27) and substituting in equation (3.8) to get

$$0 = \sum_{m=1}^{\infty} \bar{\Lambda}^{2m} \Delta_i^{(m)}(\bar{a}) + \sum_{m=1}^{\infty} \bar{\Lambda}^{2m} \beta_i^{(m)} \left(\sum_{m=0}^{\infty} \bar{\Lambda}^{2m} \Delta_k^{(m)}(\bar{a}) \right) \quad (3.28)$$

Expanding in powers of $\bar{\Lambda}^2$ in the second term and replacing the \bar{a}_k 's with a_k 's, one obtains

$$\begin{aligned} -\beta_k^{(m)}(a) &= \Delta_k^{(m)}(a) \\ &+ \sum_{n=1}^{m-1} \frac{1}{n!} \sum_{\substack{\beta_1, \dots, \beta_{n+1}=1 \\ \beta_1 + \dots + \beta_{n+1} = m}}^{n-1} \sum_{\alpha_1, \dots, \alpha_n=1}^r \left[\prod_{i=1}^n \Delta_{\alpha_i}^{(\beta_i)}(a) \right] \left(\prod_{j=1}^n \frac{\partial}{\partial a_{\alpha_j}} \right) \beta_k^{(\beta_{n+1})}(a) \end{aligned} \quad (3.29)$$

Order by order in powers of $\bar{\Lambda}^2$, the first few $\beta_k(a)$'s are

$$\begin{aligned} -\beta_k^{(1)}(a) &= \Delta_k^{(1)}(a) \\ -\beta_k^{(2)}(a) &= \Delta_k^{(2)}(a) + \sum_{l=1}^r \Delta_l^{(1)}(a) \frac{\partial \beta_k^{(1)}(a)}{\partial a_l} \\ -\beta_k^{(3)}(a) &= \Delta_k^{(3)}(a) + \sum_{l=1}^r \left[\Delta_l^{(1)}(a) \frac{\partial \beta_k^{(2)}(a)}{\partial a_l} + \Delta_l^{(2)}(a) \frac{\partial \beta_k^{(1)}(a)}{\partial a_l} \right] \\ &+ \frac{1}{2!} \sum_{l,m=1}^r \Delta_l^{(1)}(a) \Delta_m^{(1)}(a) \frac{\partial^2 \beta_k^{(1)}(a)}{\partial a_l \partial a_m} \end{aligned}$$

$$\begin{aligned}
-\beta_k^{(4)}(a) &= \Delta_k^{(4)}(a) + \sum_{l=1}^r \left[\Delta_l^{(1)}(a) \frac{\partial \beta_k^{(3)}(a)}{\partial a_l} + \Delta_l^{(2)}(a) \frac{\partial \beta_k^{(2)}(a)}{\partial a_l} \right. \\
&\quad + \left. \Delta_l^{(3)}(a) \frac{\partial \beta_k^{(1)}(a)}{\partial a_l} \right] + \frac{1}{2!} \sum_{l,m=1}^r \left[\Delta_l^{(1)}(a) \Delta_m^{(1)}(a) \frac{\partial^2 \beta_k^{(2)}(a)}{\partial a_l \partial a_m} \right. \\
&\quad + \left. 2 \Delta_l^{(1)}(a) \Delta_m^{(2)}(a) \frac{\partial^2 \beta_k^{(1)}(a)}{\partial a_l \partial a_m} \right] \\
&\quad + \frac{1}{3!} \sum_{l,m,n=1}^r \Delta_l^{(1)}(a) \Delta_m^{(1)}(a) \Delta_n^{(1)}(a) \frac{\partial^3 \beta_k^{(1)}(a)}{\partial a_l \partial a_m \partial a_n}
\end{aligned}$$

Substituting (3.29) into (3.11) reproduces the instanton corrections to the prepotential (3.13) order by order in $\bar{\Lambda}^2$.

Chapter 4

SW theory with an adjoint hypermultiplet

4.1. The Seiberg-Witten Solution for Super Yang-Mills with One Adjoint Hypermultiplet

For supersymmetric Yang-Mills theories with an asymptotic free coupling and one adjoint hypermultiplet in the adjoint representation of a classical gauge group, general arguments based on the holomorphicity of \mathcal{F} , perturbative nonrenormalization theorems beyond 1-loop order, the nature of instanton corrections, and restrictions of $U(1)_R$ invariance constrain \mathcal{F} to have the form

$$\begin{aligned} \mathcal{F}(a) = & \frac{\tau}{2} \sum_{i=1}^r a_i^2 - \frac{1}{8\pi i} \sum_{\alpha \in \mathcal{R}(\mathcal{G})} \left\{ (\alpha \cdot a)^2 \log(\alpha \cdot a)^2 \right. \\ & \left. - (\alpha \cdot a + m)^2 \log(\alpha \cdot a + m)^2 \right\} + \sum_{n=1}^{\infty} \frac{q^n}{2\pi n i} \mathcal{F}^{(n)}(a) \end{aligned} \quad (4.1)$$

where α are the roots of the gauge group \mathcal{G} . For $SU(N)$, the traceless condition $\sum_{i=1}^N a_i = 0$ is imposed.

The SW ansatz for determining the full prepotential \mathcal{F} is based on a choice of a

fibration of spectral curves over the space of vacua, and of a meromorphic 1-form $d\lambda$ on each of these curves. The renormalized order parameters a_k of the theory, their duals $a_{D,k}$, and the prepotential \mathcal{F} are given by

$$2\pi i a_k = \oint_{A_k} d\lambda, \quad 2\pi i a_{D,k} = \oint_{B_k} d\lambda, \quad a_{D,k} = \frac{\partial \mathcal{F}}{\partial a_k} \quad (4.2)$$

with A_k, B_k a suitable set of homology cycles on the spectral curves.

For $\mathcal{N} = 2$ supersymmetric gauge theories with gauge group $SU(N)$ and one hypermultiplet in the adjoint representation, a convenient parameterization for the spectral curves and meromorphic 1-forms is the Calogero-Moser case of [45]

$$f(k - \frac{m}{2}, z) = 0, \quad d\lambda = k dz \quad (4.3)$$

where

$$f(k, z) = \frac{1}{\vartheta_1(\frac{z}{2\omega_1}|\tau)} \sum_{n=0}^N \frac{1}{n!} \frac{\partial^n}{\partial z^n} \vartheta_1(\frac{z}{2\omega_1}|\tau) (-m \frac{\partial}{\partial k})^n H(k|\mathbf{k}) \quad (4.4)$$

$$H(x|\mathbf{k}) = \prod_{j=1}^N (x - k_j) \equiv (x - k_i) H_i(x|\mathbf{k}) \quad (4.5)$$

$$\vartheta_1(z|\tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{(n+1/2)^2/2} e^{2\pi i(2n+1)z} \quad (4.6)$$

along with a corresponding basis of A_k, B_k homology cycles as described in [45]. This particular choice of parameterization for the spectral curves has the geometry of a foliation over a base torus Σ , where the complex modulus τ of the torus Σ is related to the gauge coupling g and the θ -angle of the gauge theory by

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \quad (4.7)$$

Substituting (4.6) into (4.3) produces a simplified form for the spectral curves

$$\sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n(n-1)} e^{nz} H(k - mn|\mathbf{k}) = 0 \quad (4.8)$$

Substituting (4.8) into (4.2) using the Calogero-Moser parameterization (4.3) for the 1-form and performing a weak coupling expansion in powers of q similar to the methods in [45], the integral for the quantum order parameters a_i 's in terms of the classical order parameters k_i 's were calculated order by order in q producing a simplified expression of

$$a_i = k_i + \sum_{n=1}^{\infty} q^n \Delta_i^{(n)}(k)$$

where

$$\sum_{n=1}^{\infty} q^n \Delta_i^{(n)}(k) = \sum_{j=2}^{\infty} \sum_{\substack{\alpha_1, \dots, \alpha_j = -\infty, \neq 0 \\ \alpha_1 + \dots + \alpha_j = 0}} \frac{(-1)^j}{j!} \left(\frac{\partial}{\partial k_i} \right)^{j-1} \prod_{l=1}^j \left[\frac{H(k_i - \alpha_l m|\mathbf{k})}{H_i(k_i|\mathbf{k})} q^{\alpha_l^2/2} \right] \quad (4.9)$$

The first few Δ_i 's are

$$\begin{aligned} \Delta_i^{(1)}(a) &= \frac{\partial}{\partial a_i} \left[\frac{H(a_i - m|\mathbf{a})H(a_i + m|\mathbf{a})}{H_i(a_i|\mathbf{a})^2} \right] \\ \Delta_i^{(2)}(a) &= \frac{1}{4} \frac{\partial^3}{\partial a_i^3} \left[\frac{H(a_i - m|\mathbf{a})H(a_i + m|\mathbf{a})}{H_i(a_i|\mathbf{a})^2} \right]^2 \\ \Delta_i^{(3)}(a) &= \frac{1}{36} \frac{\partial^5}{\partial a_i^5} \left[\frac{H(a_i - m|\mathbf{a})H(a_i + m|\mathbf{a})}{H_i(a_i|\mathbf{a})^2} \right]^3 \\ &\quad - \frac{1}{2} \frac{\partial^2}{\partial a_i^2} \left[\frac{H(a_i - 2m|\mathbf{a})H(a_i + m|\mathbf{a})^2}{H_i(a_i|\mathbf{a})^3} \right] \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{2} \frac{\partial^2}{\partial a_i^2} \left[\frac{H(a_i - m|\mathbf{a})^2 H(a_i + 2m|\mathbf{a})}{H_i(a_i|\mathbf{a})^3} \right] \\
\Delta_i^{(4)}(a) &= \frac{1}{576} \frac{\partial^7}{\partial a_i^7} \left[\frac{H(a_i - m|\mathbf{a}) H(a_i + m|\mathbf{a})}{H_i(a_i|\mathbf{a})^2} \right]^4 \\
& + \frac{\partial}{\partial a_i} \left[\frac{H(a_i - 2m|\mathbf{a}) H(a_i + 2m|\mathbf{a})}{H_i(a_i|\mathbf{a})^2} \right] \\
& - \frac{1}{6} \frac{\partial^4}{\partial a_i^4} \left[\frac{H(a_i - 2m|\mathbf{a}) H(a_i - m|\mathbf{a}) H(a_i + m|\mathbf{a})^3}{H_i(a_i|\mathbf{a})^5} \right] \\
& - \frac{1}{6} \frac{\partial^4}{\partial a_i^4} \left[\frac{H(a_i - m|\mathbf{a})^3 H(a_i + m|\mathbf{a}) H(a_i + 2m|\mathbf{a})}{H_i(a_i|\mathbf{a})^5} \right]
\end{aligned}$$

This result can be derived in a more transparent manner by rewriting the spectral curves (4.8) as

$$k \equiv k_i + F_i(k) \quad (4.10)$$

where

$$F_i(k) = \sum_{n \in \mathbb{Z}, n \neq 0} (-1)^{n+1} q^{\frac{1}{2}n(n-1)} e^{nz} \frac{H(k - nm|\mathbf{k})}{H_i(k|\mathbf{k})} \quad (4.11)$$

An iterative solution expanded around $k = k_i$ to all orders in small q is given by

$$k = k_i + \sum_{n=1}^{\infty} y_n, \quad y_n = \frac{1}{n!} \frac{\partial^{n-1}}{\partial k^{n-1}} F_i^n(k) \big|_{k=k_i} \quad (4.12)$$

In a similar manner as in [45], z is substituted with $w = e^z$ in the the SW differential (4.3),

$$dz = \frac{dw}{w}, \quad d\lambda = k dz = k \frac{dw}{w} \quad (4.13)$$

along with the iterative solution (4.12). Performing the integral around the appo-

priate A_i cycle corresponding to k_i as prescribed in [45], reproduces (4.9) [46].

4.2. Renormalization Group Type Equations

In [45], a renormalization group type equation for the prepotential \mathcal{F} was derived

$$\frac{\partial \mathcal{F}}{\partial \tau} = \frac{1}{4\pi i} \sum_{j=1}^r \oint_{A_j} k^2 dz \quad (4.14)$$

up to an additive term independent of a_i and k_i which is physically immaterial.

Substituting the $SU(N)$ spectral curves (4.8) into (4.14) using the Calogero-Moser parameterization (4.3) for the 1-form and solving the integral in the weak coupling limit of small q gives the renormalization group like equation for the prepotential \mathcal{F} in terms of the classical order parameters k_i 's

$$\frac{\partial \mathcal{F}}{\partial \tau} = \frac{1}{2} \sum_{i=1}^r k_i^2 + \sum_{i=1}^r \sum_{n=1}^{\infty} q^n k_i \Delta_i^{(n)}(k) + \sum_{i=1}^r \sum_{n=1}^{\infty} q^n \Omega_i^{(n)}(k)$$

where

$$\sum_{n=1}^{\infty} q^n \Omega_i^{(n)}(k) = \sum_{j=2}^{\infty} \sum_{\substack{\alpha_1, \dots, \alpha_j = -\infty, \neq 0 \\ \alpha_1 + \dots + \alpha_j = 0}} \frac{(-1)^j}{j(j-2)!} \left(\frac{\partial}{\partial k_i} \right)^{j-2} \prod_{l=1}^j \left[\frac{H(k_i - \alpha_l m | \mathbf{k})}{H_i(k_i | \mathbf{k})} q^{\alpha_l^2/2} \right] \quad (4.15)$$

The first few Ω_i 's are

$$\begin{aligned} \Omega_i^{(1)}(a) &= \frac{H(a_i - m | \mathbf{a}) H(a_i + m | \mathbf{a})}{H_i(a_i | \mathbf{a})^2} \\ \Omega_i^{(2)}(a) &= \frac{3}{4} \frac{\partial^2}{\partial a_i^2} \left[\frac{H(a_i - m | \mathbf{a}) H(a_i + m | \mathbf{a})}{H_i(a_i | \mathbf{a})^2} \right]^2 \end{aligned}$$

$$\begin{aligned}
\Omega_i^{(3)}(a) &= \frac{5}{36} \frac{\partial^4}{\partial a_i^4} \left[\frac{H(a_i - m|\mathbf{a})H(a_i + m|\mathbf{a})}{H_i(a_i|\mathbf{a})^2} \right]^3 \\
&- \frac{\partial}{\partial a_i} \left[\frac{H(a_i - 2m|\mathbf{a})H(a_i + m|\mathbf{a})^2}{H_i(a_i|\mathbf{a})^3} \right] \\
&- \frac{\partial}{\partial a_i} \left[\frac{H(a_i - m|\mathbf{a})^2 H(a_i + 2m|\mathbf{a})}{H_i(a_i|\mathbf{a})^3} \right] \\
\Omega_i^{(4)}(a) &= \frac{7}{576} \frac{\partial^6}{\partial a_i^6} \left[\frac{H(a_i - m|\mathbf{a})H(a_i + m|\mathbf{a})}{H_i(a_i|\mathbf{a})^2} \right]^4 \\
&+ \frac{H(a_i - 2m|\mathbf{a})H(a_i + 2m|\mathbf{a})}{H_i(a_i|\mathbf{a})^2} \\
&- \frac{2}{3} \frac{\partial^3}{\partial a_i^3} \left[\frac{H(a_i - 2m|\mathbf{a})H(a_i - m|\mathbf{a})H(a_i + m|\mathbf{a})^3}{H_i(a_i|\mathbf{a})^5} \right] \\
&- \frac{2}{3} \frac{\partial^3}{\partial a_i^3} \left[\frac{H(a_i - m|\mathbf{a})^3 H(a_i + m|\mathbf{a})H(a_i + 2m|\mathbf{a})}{H_i(a_i|\mathbf{a})^5} \right]
\end{aligned}$$

4.3. Recursion Relations for the Prepotential \mathcal{F}

In [41], an efficient algorithm for deriving a set of recursion relations for the instanton corrections was discovered. Using similar methods as [41], a similar set of recursion relations for the prepotential \mathcal{F} was determined.

A very direct way of deriving the form of the instanton corrections to the prepotential \mathcal{F} involves substituting (4.1) and (4.9) into (4.15) to get

$$\sum_{n=1}^{\infty} q^n \mathcal{F}^{(n)}(a) = \sum_{i=1}^r \sum_{n=1}^{\infty} q^n \Omega_i^{(n)}(k) - \frac{1}{2} \sum_{i=1}^r \left[\sum_{n=1}^{\infty} q^n \Delta_i^{(n)}(k) \right]^2 \quad (4.16)$$

Then (4.9) is substituted into (4.16) and expanded in powers of q , replacing the k_i 's with a_i 's. The n -th order instanton correction to the prepotential \mathcal{F} finally takes on the form

$$\mathcal{F}^{(n)}(a) = \sum_{i=1}^r \Omega_i^{(n)}(a) - \frac{1}{2} \sum_{i=1}^r \sum_{\substack{j,l=1 \\ j+l=n}}^n \Delta_i^{(j)}(a) \Delta_i^{(l)}(a)$$

$$\begin{aligned}
& - \sum_{i=1}^{n-1} \frac{1}{i!} \sum_{\substack{\beta_1, \dots, \beta_{i+1}=1 \\ \beta_1 + \dots + \beta_{i+1} = n}}^{i-1} \sum_{\alpha_1, \dots, \alpha_i=1}^r \left[\prod_{j=1}^i \Delta_{\alpha_j}^{(\beta_j)}(a) \right] \left(\prod_{l=1}^i \frac{\partial}{\partial a_{\alpha_l}} \right) \mathcal{F}^{(\beta_{i+1})}(a)
\end{aligned} \tag{4.17}$$

The first few $\mathcal{F}^{(n)}(a)$'s are

$$\begin{aligned}
\mathcal{F}^{(1)}(a) &= \sum_{i=1}^r \Omega_i^{(1)}(a) \\
\mathcal{F}^{(2)}(a) &= \sum_{i=1}^r \Omega_i^{(2)}(a) - \frac{1}{2} \sum_{i=1}^r (\Delta_i^{(1)}(a))^2 - \sum_{i=1}^r \Delta_i^{(1)}(a) \frac{\partial \mathcal{F}^{(1)}(a)}{\partial a_i} \\
\mathcal{F}^{(3)}(a) &= \sum_{i=1}^r \Omega_i^{(3)}(a) - \frac{1}{2} \sum_{i=1}^r [2\Delta_i^{(1)}(a)\Delta_i^{(2)}(a)] \\
&\quad - \sum_{i=1}^r \left[\Delta_i^{(1)}(a) \frac{\partial \mathcal{F}^{(2)}(a)}{\partial a_i} + \Delta_i^{(2)}(a) \frac{\partial \mathcal{F}^{(1)}(a)}{\partial a_i} \right] \\
&\quad - \frac{1}{2!} \sum_{i,j=1}^r \Delta_i^{(1)}(a)\Delta_j^{(1)}(a) \frac{\partial^2 \mathcal{F}^{(1)}(a)}{\partial a_i \partial a_j} \\
\mathcal{F}^{(4)}(a) &= \sum_{i=1}^r \Omega_i^{(4)}(a) - \frac{1}{2} \sum_{i=1}^r [2\Delta_i^{(1)}(a)\Delta_i^{(3)}(a) + (\Delta_i^{(2)}(a))^2] \\
&\quad - \sum_{i=1}^r \left[\Delta_i^{(1)}(a) \frac{\partial \mathcal{F}^{(3)}(a)}{\partial a_i} + \Delta_i^{(2)}(a) \frac{\partial \mathcal{F}^{(2)}(a)}{\partial a_i} + \Delta_i^{(3)}(a) \frac{\partial \mathcal{F}^{(1)}(a)}{\partial a_i} \right] \\
&\quad - \frac{1}{2!} \sum_{i,j=1}^r \left[\Delta_i^{(1)}(a)\Delta_j^{(1)}(a) \frac{\partial^2 \mathcal{F}^{(2)}(a)}{\partial a_i \partial a_j} + 2\Delta_i^{(1)}(a)\Delta_j^{(2)}(a) \frac{\partial^2 \mathcal{F}^{(1)}(a)}{\partial a_i \partial a_j} \right] \\
&\quad - \frac{1}{3!} \sum_{i,j,k=1}^r \Delta_i^{(1)}(a)\Delta_j^{(1)}(a)\Delta_k^{(1)}(a) \frac{\partial^3 \mathcal{F}^{(1)}(a)}{\partial a_i \partial a_j \partial a_k}
\end{aligned} \tag{4.18}$$

4.4. Comparison With Previous Results

In the limit the full hypermultiplet is decoupled with $\tau \rightarrow \infty, m \rightarrow \infty$ while keeping constant the parameters k_i and Λ :

$$\Lambda^{2N} = (-1)^N m^{2N} q \quad q = e^{2\pi i \tau} \tag{4.19}$$

equations (4.9) and (4.15) break down to their corresponding equations in the pure $SU(N)$ gauge theory cases [40][41][35].

In the $\mathcal{N}=4$ limit where $m \rightarrow 0$, all the Ω_i and Δ_i terms in (4.9) and (4.15) vanish and reproduces the expected prepotential

$$\mathcal{F}(a) = \frac{\tau}{2} \sum_{i=1}^N a_i^2 \quad (4.20)$$

for $SU(N)$.

For $SU(2)$, the existing results in the literature have the instanton expressed in term of

$$a_1 = a, \quad a_2 = -a \quad (4.21)$$

Explicit evaluations of the first three instanton corrections are

$$\begin{aligned} \mathcal{F}^{(1)}(a) &= \frac{m^4}{2a^2} \\ \frac{\mathcal{F}^{(2)}(a)}{2} &= -\frac{9m^6}{16a^4} + \frac{5m^8}{64a^6} \\ \frac{\mathcal{F}^{(3)}(a)}{3} &= \frac{m^6}{a^4} + \frac{25m^8}{48a^6} - \frac{67m^{10}}{192a^8} + \frac{3m^{12}}{64a^{10}} \end{aligned} \quad (4.22)$$

which disagree with results in the literature beyond one instanton [48]¹, but agrees in the limit where the full hypermultiplet decouples [41]. It turns out that performing the Seiberg-Witten elliptic function calculation in [47] to higher instanton orders reproduces the instanton calculations of [48].

¹In the limit of decoupling the full $SU(2)$ adjoint hypermultiplet in [48], there is a discrepancy of a factor $\frac{1}{2}$ with the pure $SU(2)$ results of [21][41].

On the other hand, the $SU(2)$ spectral curve from the Calogero-Moser construction (4.3) can be explicitly shown to be equivalent to the $SU(2)$ mass deformed $\mathcal{N}=4$ spectral curve construction [2] up to reparameterizations of the classical order parameters k_i 's. This spectral curve forms a crucial part of the elliptic function calculation in [47].

One possible problem with the elliptic function calculation in [47] is the assumption of Matone's relation [23][34] holding in the presence of an adjoint hypermultiplet. Generalizations of Matone's relation for classes of $\mathcal{N}=2$ SUSY gauge theories with fundamental hypermultiplets was proven in general [35] and corresponds to a renormalization group type of equation for the prepotential \mathcal{F} . For the case of one adjoint hypermultiplet, a renormalization group equation for the prepotential \mathcal{F} was derived [45] and calculated to all orders (4.15) which differs greatly from Matone's relation and [34][35], but agrees with the latter cases in the limit the full adjoint hypermultiplet is decoupled [45].

4.5. S-Duality Properties

A closer examination of the Calogero-Moser parameterization of the Seiberg-Witten spectral curves and 1-form (4.3) and (4.5) reveals there's an implicit S-duality present.

Using the transformation property of the theta functions

$$\vartheta_1\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right) = \sqrt{-i\tau} \exp\left(\frac{iz^2}{\pi\tau}\right) \vartheta_1(z|\tau) \quad (4.23)$$

and substituting it into (4.3) and (4.5) shows explicitly that the form of the spectral curves and 1-form are indeed invariant under S-duality transformations up to reparameterizations of the classical order parameters k_i 's. Correspondingly, the roles

of the A and B cycles in the Seiberg-Witten ansatz (4.2) are interchanged under S-duality transformations.

With this explicit S-duality, the corresponding weakly coupled "dual" theory in the magnetic sector of the theory expanded around a small "dual" coupling constant

$$q_D = e^{2\pi i \tau_D} \quad \tau_D = -\frac{1}{\tau} \quad \tau \rightarrow i0^+ \quad (4.24)$$

will have a corresponding "dual" prepotential $\mathcal{F}_D(a_D)$ identical in form to the prepotential $\mathcal{F}(a)$ in (4.1) and (4.17) with the corresponding substitutions of the coupling constant and quantum order parameters to their "dual" counterparts

$$q \rightarrow q_D \quad a_i \rightarrow a_{D,i} \quad (4.25)$$

respectively. This can be interpreted as a non-perturbative expansion of the theory, where the dynamics of the strongly coupled regime in the electric sector of the theory is described by the dynamics of a corresponding weakly coupled "dual" theory in the magnetic sector of the same theory. (Other strong coupling expansions in the same spirit were performed in [21][49][53]).

Considering there are claims that the Calogero-Moser system can be constructed explicitly from the Hitchin system [44], this S-duality is like a realization of the Donagi-Witten construction of Seiberg-Witten theory using the Hitchin system [42] where an underlying S-duality and general $SL(2, Z)$ symmetry is built into the geometry of the foliation over a base torus Σ construction (4.7) from the start. In the prepotential calculations performed around small coupling q or q_D , the S-duality is explicitly broken while the underlying spectral curve (4.3) is invariant under S-duality and in general an $SL(2, Z)$ symmetry [2][42].

4.6. Generalizations to Other Gauge Groups

Generalizations of the $SU(N)$ Calogero-Moser integrable system were investigated in [50][51] for various cases of twisted and untwisted gauge groups, but stopped short of producing parameterizations suitable for use as Seiberg-Witten spectral curves. Possible parameterizations to general untwisted classical gauge groups can be conjectured starting from the $SU(N)$ spectral curves and placing appropriate constraints such that decouplings of the full adjoint hypermultiplet reproduce the pure gauge theory results [7][6][13][52].

In the spirit of [13][52], one possibility is to replace the $H(k)$ polynomial with

$$H(x|\mathbf{k}) \rightarrow H(x|\mathbf{k}) = \prod_{j=1}^N (x^2 - k_j^2) \equiv (x - k_i)(x + k_i)H_i(x|\mathbf{k}) \quad (4.26)$$

in the Calogero-Moser parameterization of the SW spectral curves (4.8).

The appropriate limits for full hypermultiplet decoupling are $\tau \rightarrow \infty, m \rightarrow \infty$ while keeping constant the parameters k_i and Λ :

$$\begin{aligned} SO(2r) \quad \Lambda^{4r-4} &\equiv m^{4r-4}q \\ SO(2r+1) \quad \Lambda^{4r-2} &\equiv m^{4r-2}q \\ SO(2r) \quad \Lambda^{4r+4} &\equiv m^{4r+4}q \end{aligned} \quad (4.27)$$

where $q = e^{2\pi i\tau}$.

Chapter 5

Conclusion

The recursion relations discovered [41][56] improve considerably the ability to evaluate explicitly the non-perturbative instanton corrections to $\mathcal{N} = 2$ super Yang-Mills theories. One may speculate as to the existence of similar recursion relations for the strong coupling problem [21][49] in the absence of an explicit S-duality. Other possible directions would be in speculating on the existence of simple recursion relations in the mirror symmetry problems of calculating worldsheet instanton corrections to string theory [59].

A deeper question is that of the existence of the recursion relations in the first place. One approach is that the instanton corrections to the prepotential of $\mathcal{N} = 2$ super Yang-Mills is related to a topological field theory version of the same theory where the the Lorentz group and \mathcal{R} -symmetry are twisted into one another [60][61]. Another question is that of the existence of the role of integrable systems in the spectral curves, and how it relates to topological field theory and D-branes [62].

One is left to wonder if this "simplicity" that appears in Seiberg-Witten theory is just the tip of the iceberg for even more exact calculations in nonperturbative string theories.

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